Math 210C Lecture 25 Notes

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1 Characters of Representations

1.1 Definition and examples

Recall that if $A, B \in M_n(R)$, then $\mathrm{tr}(AB) - \mathrm{tr}(BA)$. In particular, if $\varphi \in Aut_F(V)$, where F is a field and V is a finite dimensional vector space over F, then $\mathrm{tr}(\varphi)$ is defined independently of and does not depend on the choice of basis.

Definition 1.1. Let V be a finite dimensional F-vector space. The **character** of a representation $\rho : G \to \mathrm{Aut}_F(V)$ is $\chi_V : G \to D$, given by $\chi_V(g) = \mathrm{tr}(\rho(g))$. We say that χ_V is a **character of** G.

Definition 1.2. A character χ is **irreducible** if there exists an irreducible representation V such that $\chi = \chi_V$.

Example 1.1. If $\rho : G \to F^\times$ is a homomrphism (i.e. the image is a 1×1 matrix), then we get an **abelian character** $\chi : G \to F$ given by $g \mapsto \rho(g)$.

Example 1.2. Let $\rho : S_n \to \mathrm{GL}_n(F)$ be the permutation representation. Then $\chi(\sigma) = |[n]^\sigma|$, the number of fixed points under σ.

Example 1.3. Let W be the 2 dimensional representation of S_3. Then if $\tau = (1\,2)$ and $\sigma = (1\,2\,3)$,

$$\rho_W(\sigma) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \rho_W(\tau) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

So we get $\chi_W(\sigma) = -1$, $\chi_W(\tau) = 0$, and $\chi_W(e) = 2$.

Example 1.4. Let $F[G]$ have the regular representation: $g : F[G] \to F[G]$ sends $h \mapsto gh$. Then

$$\chi_{F[G]}(g) = \begin{cases} 0 & g \neq 1 \\ |G| & g = 1. \end{cases}$$

Definition 1.3. The character of the trivial representation (every $\rho(g) = 1$ for all g) is te **trivial character** or (**principal character**).

1.2 Characters as class functions

Definition 1.4. A **class function** $f : G \to F$ is a function that is constant on conjugacy classes.

Lemma 1.1. *Let V, W be finite dimensional F-representations of G.*

1. $\chi_V(e) = \dim(V)$.
2. $\chi_V = \chi_W$ *if* $V \cong W$
3. $\chi_{V \oplus W} = \chi_V + \chi_W$.
4. χ_V *is a class function.*

Proof. For the first statement, the matrix $\rho_V(e) = I_n$, which has trace $\dim(V)$.

For the second statement $\rho_V(g)$ and $\rho_W(g)$ will be the same matrix but in a different basis.

For the third statement, we have th block matrix

$$\rho_{V \oplus W} = \begin{bmatrix} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{bmatrix},$$

so $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$.

For the fourth statement,

$$\chi_V(hgh^{-1}) = \operatorname{tr}(\rho_V(hgh^1)) = \operatorname{tr}(\rho_V(h)\rho_V(g)\rho_V(h)^{-1}) = \operatorname{tr}(\rho_V(g)) = \chi_V(g). \qquad \square$$

Proposition 1.1. *Let G be a finite group, let F be algebraically closed, and let* $\operatorname{char}(F) \nmid |G|$. *Then the irredicible F- characters of G form a basis of the F-vector space of class functions on G.*

Proof. Let $g_1, \dots, g_n$ be representatives of the conjugacy classes in G, and let $\chi_1, \dots, \chi_r$ be the irreducible characters of G. We claim that $\chi_1, \dots, \chi_r$ are linearly independent, which implies the result. Write $\chi_i = \chi_{V_i}$, where V_i is irreducible and $n_i = \dim_F(V_i)$. Then

$$F[G] \cong \prod_{i=1}^{r} M_{n_i}(F) \cong V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}.$$

Now observe that χ_i extends to $\chi_i : F[G] \to F$ F-linearly: $\sum_g a_g g \mapsto \sum_g a_g \chi_i(g)$. If $x \in F[G]$, then we get$x : V_i \to V_i$, so $\chi_i(x) = \operatorname{tr}(x : V_i \to V_i)$. We have idempotents $e_i : V_i \xrightarrow{\text{id}} V_i$ and $e_i : V_j \xrightarrow{0} V_j$ for $j \neq i$. Then

$$\chi_i(e_j) = \begin{cases} n_i & i = j \\ 0 & i \neq j.. \end{cases}$$

So if $\phi = \sum_{i=1}^{r} a_i \chi_i = 0$, then $|phii(e_j) = a_j n_j = 0$, so $a_j = 0$ for all j. So the χ_i are F-linearly independent. $\square$

Lemma 1.2. *Let G be finite, let $\text{char}(F) = 0$, and let V and W be finite dimensional F-vector spaces. Then $V \cong W$ if and only if $\chi_V = \chi_W$.*

Proof. We may replace F by its algebraic closure. If $V \cong_{\overline{F}} W$, then $V \otimes_F \overline{F} \cong W \otimes_F \overline{F}$. Since tensoring with $\overline{F}$ is faithfully flat, $V \cong W$.

$F[G] = \prod_{i=1}^k M_{n_i}(F)$, where $e_i \in M_{n_i}(F)$. We have $V = V_1^{m_1} \oplus \cdots \oplus V_r^{m_r}$. Then $\chi_V(e_i) = m_i \dim_F(V_i)$. This determines each m_i from χ_V, so χ_V determines the isomorphism class of V. $\square$

1.3 Projections onto irreducible representations

Definition 1.5. The **degree** of a character χ_i is $n_i = \dim_F(V_i)$, where $\chi_i = \chi_{V_i}$.

Proposition 1.2. *Let G be finite, let F be algebraically closed, and let $\text{char}(F) \nmid |G|$. Let $\chi_1, \dots, \chi_r$ be the irreducible characters of G, and let $n_i = \deg(\chi_i)$. Then*

$$e_i = \frac{n_i}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g$$

are the central primitive orthogonal idempotents of $F[G]$.

Proof. Let f_i be the i-th primitive orthogonal central idempotent in $F[G]$ corresponding to χ_i. Then $f_i = \sum_[g \in G] a_g g \in F[G]$. For $g \in G$,

$$\chi_{F[G]}(f_i g^{-1}) = \sum_{h \in G} \chi_{F[G]}(a_h h g^{-1}) = a_g |G|.$$

On the other hand, $\chi_{F[G]} = \sum_{i=1}^r n_i \chi_i$, so

$$\chi_{F[G]}(f_i g^{-1}) = \sum_{j=1}^r n_j \chi_j (f_i g^{-1}).$$

We can extend ρ_{V_j} to $\rho : F[G] \to \text{End}_F(V_j)$, given by extending ρ_{V_j} F-linearly. Then

$$\rho_j(f_i g^{-1}) = \rho_j(f_i) \rho_j(g^{-1}) = \delta_{i,j} \rho_j(g^{-1}).$$

So $\chi_j(f_i g^{-1}) = \delta_{i,j} \chi_j(g^{-1})$. So we get

$$\chi_{F[G]}(g_i g^{-1}) = \sum_{j=1}^r n_i \delta_{i,j} \chi_j(g^{-1}) = n_i \chi_i(g^{-1}) = a_g |G|.$$

So we get

$$a_g = \frac{n_i}{|G|} \chi_i(g^{-1}).$$

Plugging these coefficients back into the original expression gives the result. $\square$

Corollary 1.1. *Let n_i be as above. Then $n_i \neq 0$ in F.*

Lemma 1.3. *Let V be an F-representation of G, and let E/F be a field extension. Then $E \otimes_F V$ is an $E[G]$-module via $g(\alpha \otimes v) = \alpha \otimes gv$.*

Proof. $E[G] = E \otimes_F F[G] \circlearrowright E \otimes_F V$ coordinate-wise. $\square$

Lemma 1.4. *Let G be finite, let F be algebraically closed, and let* $\mathrm{char}(F) \nmid |G|$. *Let V be a finite dimensional F-representation of G. Then for all $g \in G$, the automorphism $\rho_V(g)$ is diagonalizable.*

Proof. Fix $g \in G$. We can replace G by $\langle g \rangle$. Then $F[G] \bigoplus_{i=1}^{|G|} V_i$, where these are 1-dimensional representations. Choose a basis coming from each of these. $\square$